1. Want to derive

$$\rho \frac{d\vec{V}}{dt} = -\nabla P + \frac{1}{c}(\vec{j} \times \vec{B}) \qquad (13.3\ G\&L)$$

from $\rho_j [(\vec{V_j} \cdot \nabla)\vec{V_j} + \frac{\partial \vec{V_j}}{\partial t}] = \rho_j^c \vec{E} + \rho_j^c (\vec{V_j} \times \vec{B}) - \nabla P_j \qquad (6.25)$

mass density

charge density $\rho_j^c = n_j q_j$

① $(\vec{V_j} \cdot \nabla)\vec{V_j} + \frac{\partial \vec{V_j}}{\partial t} = \frac{d\vec{V_j}}{dt}$.

② for ions and electrons, (6.25) becomes

$$\rho_i \cdot \frac{d\vec{V_i}}{dt} = n_i q_i \vec{E} + n_i q_i (\vec{V_i} \times \vec{B}) - \nabla P_i$$

$$\rho_e \frac{d\vec{V_e}}{dt} = n_e q_e \vec{E} + n_e q_e (\vec{V_e} \times \vec{B}) - \nabla P_e$$

Add these two equ. together, set

$$\rho_i \frac{d\vec{V_i}}{dt} + \rho_e \frac{d\vec{V_e}}{dt} = \underbrace{(n_i q_i + n_e q_e)}_{0} \vec{E} + (n_i q_i \vec{V_i} + n_e q_e \vec{V_e}) \times \vec{B} - (\nabla P_i + \nabla P_e)$$

0, due to neutrality in plasma.

$\vec{j} = n_i q_i \vec{V_i} + n_e q_e \vec{V_e}$.

$\nabla P = \nabla P_i + \nabla P_e$.

Also, let $\rho_i \frac{d\vec{V_i}}{dt} + \rho_e \frac{d\vec{V_e}}{dt} = \rho \frac{d\vec{V}}{dt}$.

$$\Rightarrow \rho \frac{d\vec{V}}{dt} = \vec{j} \times \vec{B} - \nabla P$$

This is equivalent to (13.3 G&L), $\frac{1}{c}$ is due to different unit system

~~5~~ 2. To derive $p + \frac{B^2}{2\mu_0} = \text{constant}$.

(13.2 G&L) at steady state.

$$-\nabla p + \vec{j} \times \vec{B} = 0$$

Maxwell equation. $\nabla \times \vec{B} = \mu_0 \vec{J} + \cancel{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}} \to 0$ due to steady state.

$$\Rightarrow \nabla p = \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B}$$

With $(\nabla \times \vec{B}) \times \vec{B} = \underbrace{\cancel{(\vec{B} \cdot \nabla)\vec{B}}}_{0} - \nabla (\vec{B} \cdot \vec{B})$

$$= -\frac{1}{2} \nabla B^2$$

We get $\nabla p = -\frac{1}{2\mu_0} \nabla B^2$

$$\nabla (p + \frac{1}{2\mu_0} B^2) = 0.$$

$$p + \frac{B^2}{2\mu_0} = \text{Constant}.$$

$$= \frac{B_0^2}{2\mu_0}$$

3. Refer to Pg. 488 – Pg. 490

$$0 = -\nabla p_0 + \frac{1}{c} (\vec{j}_0 \times \vec{B}_0)$$

With linear perturbation. $p = p_0 + p'$, $\vec{j} = \vec{j}_0 + \vec{j}'$, $\vec{B} = \vec{B}_0 + \vec{B}'$

It becomes $-\nabla p' + \frac{1}{c} (\vec{j}' \times \vec{B}_0 + \vec{j}_0 \times \vec{B}') = 0$ ①

(13.5) $\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} \Rightarrow \begin{cases} \nabla \times \vec{B}_0 = \frac{4\pi}{c} \vec{j}_0 & ② \\ \nabla \times \vec{B}' = \frac{4\pi}{c} \vec{j}' & ③ \end{cases}$

Also, $\vec{B}' = \nabla \times (\vec{\xi} \times \vec{B}_0)$ ——— ④

$\vec{p}' = -\gamma p_0 \nabla \cdot \vec{\xi} - \vec{\xi} \cdot \nabla \vec{p}_0$ ——— ⑤

$$\Rightarrow 0 = \nabla (\gamma p_0 \nabla \cdot \vec{\xi} + \vec{\xi} \cdot \nabla p_0] + \frac{1}{4\pi} [\nabla \times \nabla \times (\vec{\xi} \times \vec{B}_0) \times \vec{B}_0 + \nabla \times \vec{B}_0 \times \nabla \times (\vec{\xi} \times \vec{B}_0)]$$

4. 13.17 − 13.18

$$\delta W = \int_V \Big[\underset{\overset{\uparrow}{A}}{\frac{Q^2}{4\pi}} - \underset{\overset{\uparrow}{A}}{\vec{j}\cdot Q\times\vec{\xi}} + \underset{\overset{\uparrow}{B}}{\gamma P_0 (\nabla\cdot\vec{\xi})^2} + \underset{\overset{\uparrow}{B}}{(\nabla\cdot\vec{\xi})(\vec{\xi}\cdot\nabla p)} \Big] d^3x$$

$$- \frac{1}{2}\int_S (\vec{n}\cdot\vec{\xi})\Big[\underset{\overset{\uparrow}{B}}{\gamma P_0 \nabla\cdot\vec{\xi}} + \underset{\overset{\uparrow}{B}}{\vec{\xi}\cdot\nabla P_0} - \underset{\overset{\uparrow}{A}}{\frac{1}{4\pi}\vec{B}_0\cdot Q}\Big] ds$$

Here, A means "current driven" term

B means " pressure driven" term.

($Q \equiv \nabla\times(\vec{\xi}\times\vec{B}_0) = \vec{B}'$, is related to $\vec{j}'$)

In stellartor, B field shear is not achieved using current, so there is no "current-driving" instability. In this sense, stellarator is more stable than ~~Tokm~~ Tokamak.

5. (a) $b_i = \dfrac{B_z\,(int)}{B_\theta}$, $B_z\,(ext) = 0$ } In this case, the internal B field equals to $B_z\,(int)$ and the external B field equals to B_θ.

Pressure balance $P_{kin} + P_{int} = P_{ext}$.

$$2nkT + \frac{B_z^2(int)}{2\mu} = \frac{B_\theta^2}{2\mu}.$$

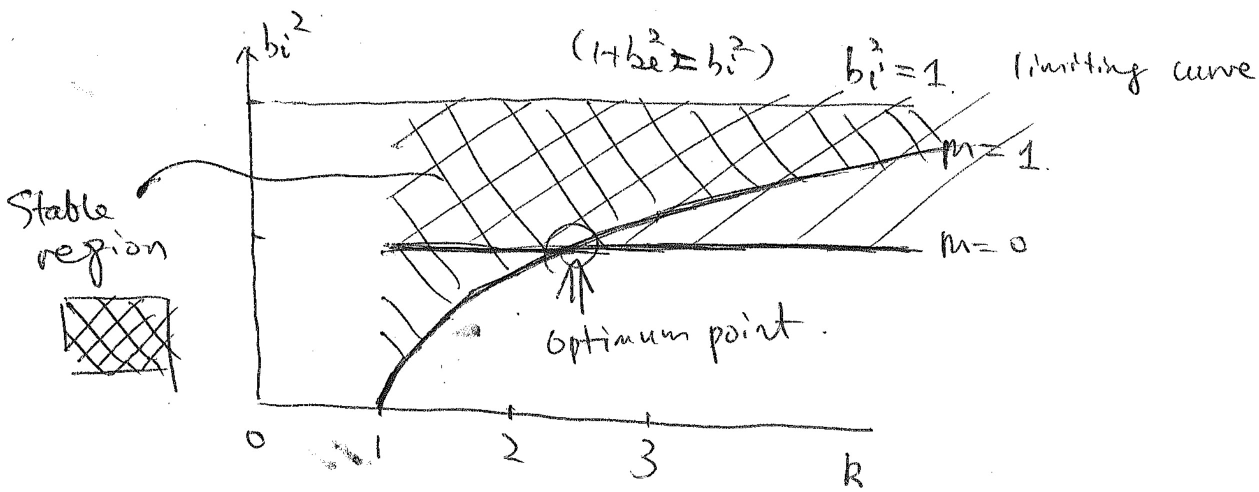
$$\text{So, } b_i^2 = \frac{B_z^2\,(int)}{B_\theta^2} = 1 - \frac{2nkT}{B_\theta^2/2\mu} = 1 - \beta$$

(b). Based on Fig. 13.2

For $B_z(ext) = 0$, i.e, $b_e = 0$

	$k=1$	$k=1.5$	$k=2$	$k=3$	$k=\infty$
b_i (m=1)	0	0.5	0.6	0.75	1
b_i^2 (m=1)	0	0.25	0.36	0.56	1

	$k=1$	$k=1.5$	$k=2$	$k=3$	$k=\infty$
b_i (m=0)	0.7	0.7	0.7	0.7	0.7
b_i^2	0.49	0.49	0.49	0.49	0.49



Stable within the area b/w the curve $\binom{m=1}{m=0}$ and the limiting curve $b_i^2 = 1$.

(c) Optimum operation requires highest β and highest k.

lowest b_i^2, so $b_i^2 = 0.49$. $\Leftarrow b_i^2 = 1-\beta$ ↳ to give high power density.

So, the optimum point is found as shown above.

⑥ A) $-\rho\Omega^2 \xi = \nabla(\gamma p \nabla\cdot\xi) - \frac{1}{4\pi} B\times(\nabla\times Q)$ where $\xi = [\xi_r(r)\,\xi_\theta(r)\,\xi_z(r)]\exp(im\theta + ikz)$

From variational method and δW_f:
$$Q_r = ikB_z\xi_r$$
$$Q_\theta = ikB_z\xi_\theta$$
$$Q_z = -B_z\left[\frac{1}{r}\partial_r(r\xi_r) + \frac{im}{r}\xi_\theta\right]$$

The above equation can then be separated into:
$$-\rho\Omega^2\xi_r = \partial_r(\gamma p\nabla\cdot\xi) - \frac{B_z}{4\pi}(\partial_r Q_z - ikQ_r)$$
$$-\rho\Omega^2\xi_\theta = \frac{im}{r}(\gamma p\nabla\cdot\xi) - \frac{B_z}{4\pi}\left(\frac{im}{r}Q_z - ikQ_\theta\right)$$
$$-\rho\Omega^2\xi_z = ik(\gamma p\nabla\cdot\xi)$$

$$\psi = \gamma p\nabla\cdot\xi - \frac{1}{4\pi}B\cdot Q = \left(\gamma p + \frac{B_z^2}{4\pi}\right)\left[\frac{1}{r}\partial_r(r\xi_r) + \frac{m}{r}\xi_\theta\right) + ik\gamma p\,\xi_z$$

The above terms can then be rewritten with ψ as:
$$\left(\frac{k^2B_z^2}{4\pi} - \rho\Omega^2\right)\xi_r = \partial_r\psi$$
$$\left(\frac{k^2B_z^2}{4\pi} - \rho\Omega^2\right)\xi_\theta = \frac{im}{r}\psi$$
$$\left[\frac{k^2B_z^2}{4\pi} - \rho\Omega^2\left(1 + \frac{B_z^2}{4\pi\gamma p}\right)\right]\xi_r = ik\psi$$

These values can be substituted into the above equation for ψ which results in:

$$\boxed{\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\psi}{\partial r} - \left(k^2 + \frac{m^2}{r^2}\right)\psi = 0 \qquad \text{where } K^2 = \frac{\frac{k^2B_z^2}{4\pi} - \rho\Omega^2}{\gamma p + B_z^2/4\pi}\left\{1 + \frac{k^2\gamma p}{\frac{k^2B_z^2}{4\pi} - \rho\Omega^2\left(1 + \frac{B_z^2}{4\pi\gamma p}\right)}\right\}}$$

B) $r^2 \frac{\partial^2 \psi}{\partial r^2} + r \frac{\partial \psi}{\partial r} - K^2 r^2 \psi - m^2 \psi = 0$

This is modified Bessel's equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+\alpha^2) y = 0$$

$$\boxed{\psi = c_1 I_m(Kr) + c_2 K_m(Kr)}$$

7) $\mu = \frac{B_\theta}{rB_z}$. $\frac{r}{4}\left(\frac{1}{\mu}\cdot\frac{\partial \mu}{\partial r}\right)^2 + \frac{8\pi}{B_z^2}\left(\frac{\partial p}{\partial r}\right) > 0$

$\frac{\partial \mu}{\partial r} = -\frac{1}{r^2}\cdot\frac{B_\theta}{B_z}$. $\frac{1}{\mu} = \frac{rB_z}{B_\theta}$

Shear : $\frac{1}{\mu}\cdot\frac{\partial \mu}{\partial r} = \frac{\cancel{r}\cancel{B_z}}{\cancel{B_\theta}}\cdot -\frac{1}{r\cancel{^2}}\cdot\frac{\cancel{B_\theta}}{\cancel{B_z}}$ so Shear $= \left|-\frac{1}{r}\right|$

At $r=0$, shear $\to \infty$

At $r=a$, shear $= \left|-\frac{1}{a}\right|$

- Shear is smaller on the outer radius which makes sense because the radius is larger so the shear is less.